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# Toroidal groups without non-constant meromorphic functions

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## 1 Introduction

Gheradelli and Andreotti [4] obtained fibration theorem for quasi-Abelian varieties. Abe [1] proved the fibration theorems by getting some standard forms of period matrices. Umeno [8] characterized the quasi-Abelian varieties by the above standard forms of period matrices.

We shall study meromorphic functions on a toroidal group by using period matrices for quasi-Abelian varieties. In the section 2, we shall give the conditions that a toroidal group has no non-constant meromorphic functions. In the section 3, we shall discuss the example given by Abe and Kopfermann using the recent results of Umeno [8].

## 2 Meromorphic functions on a toroidal group

In this section, we discuss meromorphic functions on a toroidal group. Before proceeding, we introduce some definitions and terminologies.

A connected complex Lie group  $X$  is called a toroidal group if every holomorphic function on  $X$  is constant.

Since any toroidal group is an abelian Lie group, there exists a discrete subgroup  $\Gamma$  of  $\mathbb{C}^n$  such that  $X$  is isomorphic to  $\mathbb{C}^n/\Gamma$ . Let  $X = \mathbb{C}^n/\Gamma$  be a toroidal group and  $\Gamma = \mathbb{Z}\{\lambda_1, \dots, \lambda_{n+q}\}$ ,  $0 < q \leq n$  be a discrete subgroup of  $\mathbb{C}^n$  generated by  $\mathbb{R}$ -linearly independent vectors  $\lambda_1, \dots, \lambda_{n+q}$ . The matrix  $P = [\lambda_1, \dots, \lambda_{n+q}]$  is called a period matrix for  $X = \mathbb{C}^n/\Gamma$ . We sometimes write  $\Gamma = \mathbb{Z}\{P\}$  instead of  $\Gamma = \mathbb{Z}\{\lambda_1, \dots, \lambda_{n+q}\}$ . Let  $\mathbb{R}_\Gamma = \mathbb{R}\{\lambda_1, \dots, \lambda_{n+q}\}$  be the  $\mathbb{R}$ -span of  $\Gamma$ . We denote by  $\mathbb{C}_\Gamma = \mathbb{R}_\Gamma \cap \sqrt{-1}\mathbb{R}_\Gamma$  the maximal complex subspace of  $\mathbb{R}_\Gamma$ .

**Definition 2.1** A toroidal group  $\mathbb{C}^n/\Gamma$  is of type  $q$  ( $q > 0$ ) if

$$\dim_{\mathbb{C}} \mathbb{C}_{\Gamma} = q.$$

**Definition 2.2** A toroidal group  $\mathbb{C}^n/\Gamma$  is a quasi-Abelian variety, if there exists a Hermitian form  $H$  on  $\mathbb{C}^n \times \mathbb{C}^n$  such that

$$H \mid \mathbb{C}_{\Gamma} \times \mathbb{C}_{\Gamma} > 0 \text{ and}$$

$$E := \text{Im } H \mid \Gamma \times \Gamma \text{ is a } \mathbb{Z}\text{-valued skew-symmetric form.}$$

A Hermitian form  $H$  is called an ample Riemann form which defines a quasi-Abelian structure on  $X = \mathbb{C}^n/\Gamma$ . Let  $f(z)$  be a meromorphic function on  $\mathbb{C}^n$ . A period of  $f$  is a vector  $\lambda \in \mathbb{C}^n$  such that  $f(z + \lambda) = f(z)$  for all  $z \in \mathbb{C}^n$  and the period group of  $f$  is the set  $G(f)$  of all periods of  $f$ .

For later use, we first consider the following([3]):

**Theorem 2.1** *Let  $X = \mathbb{C}^n/\Gamma$  be a toroidal group and  $f$  be a meromorphic function on  $\mathbb{C}^n$  with  $\Gamma \subset G(f)$ . Then there exist  $p, q \in H^0(\mathbb{C}^n, \mathcal{O})$  with  $(p, q) = 1$  and  $f = p/q$ , and there exist linear polynomials  $l_{\lambda} (\lambda \in \Gamma)$  such that*

$$\begin{aligned} p(z + \lambda) &= p(z) \exp(l_{\lambda}(z)) \text{ and} \\ q(z + \lambda) &= q(z) \exp(l_{\lambda}(z)), \end{aligned}$$

for all  $z \in \mathbb{C}^n$  and  $\lambda \in \Gamma$ .

Next, let us set  $el_{\lambda}(z) := \exp(l_{\lambda}(z))$ . Then we see

$$el_{\lambda'}(z + \lambda)el_{\lambda}(z) = el_{\lambda}(z + \lambda')el_{\lambda'}(z),$$

since  $el_{\lambda'+\lambda}(z) = el_{\lambda'}(z + \lambda)el_{\lambda}(z)$ .

**Definition 2.3** A system of holomorphic functions  $e_{\lambda} \in H^0(\mathbb{C}^n, \mathcal{O}^*)$  satisfying

$$e_{\lambda'}(z + \lambda)e_{\lambda}(z) = e_{\lambda}(z + \lambda')e_{\lambda'}(z)$$

is said to be multipliers.

We have already known the following(cf.[6]):

**Proposition 2.1** *Let  $X = \mathbb{C}^n/\Gamma$  be a toroidal group and  $L \rightarrow X$  be a complex line bundle. Then, for each  $\lambda \in \Gamma$ , there exist multipliers  $e_\lambda$  such that*

$$L \cong \mathbb{C}^n \times \mathbb{C}/\Gamma$$

where  $\Gamma$  acts on  $\mathbb{C}^n \times \mathbb{C}$  by  $\lambda \circ (z, \xi) = (z + \lambda, e_\lambda(z)\xi)$  for  $\lambda \in \Gamma$ .

Set  $e_\lambda(z) = \exp(2\pi\sqrt{-1}f_\lambda(z))$  where  $f_\lambda \in H^0(\mathbb{C}^n, \mathcal{O})$ . For the line bundle  $L$  defined by  $e_\lambda \in H^0(\mathbb{C}^n, \mathcal{O}^*)$ , we see the following([6]):

**Proposition 2.2** *Let  $L$  be a line bundle on a toroidal group  $X = \mathbb{C}^n/\Gamma$  defined by  $e_\lambda(z) = \exp(2\pi\sqrt{-1}f_\lambda(z))$  such that  $c_1(L) = E$ . Then*

$$E(\lambda_1, \lambda_2) = f_{\lambda_2}(z + \lambda_1) + f_{\lambda_1}(z) - f_{\lambda_1}(z + \lambda_2) - f_{\lambda_2}(z) \text{ for } z \in \mathbb{C}^n, \text{ and } \lambda_i \in \Gamma$$

Here, we recall the definition of Néron-Severi group of  $X$ . Let  $X$  be a toroidal group. The Néron-Severi group  $NS(X)$  of  $X$  is defined by

$$NS(X) = \{E : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{R} \mid E : \text{an alternating form with } E(\Gamma \times \Gamma) \subseteq \mathbb{Z} \text{ and } E(\sqrt{-1}\lambda, \sqrt{-1}\mu) = E(\lambda, \mu)\}.$$

**Definition 2.4** A toroidal group  $\mathbb{C}^n/\Gamma$  is called of cohomologically finite type if

$$\dim H^1(\mathbb{C}^n/\Gamma, \mathcal{O}) < +\infty$$

Now, we state our main theorem.

**Theorem 2.2** *Let  $X = \mathbb{C}^n/\Gamma$  be a toroidal group of cohomologically finite type. Suppose that the Néron-Severi group  $NS(X)$  is zero. Then  $X$  has no non-constant meromorphic functions.*

To prove theorem 2.2, we need some results. So, we first consider the following result well known in classical complex torus theory such as Appell-Humbert decomposition [7].

**Theorem 2.3** *Let  $X = \mathbb{C}^n/\Gamma$  be a toroidal group,  $L \rightarrow X$  a complex line bundle such that  $c_1(L) = E \in H^2(X, \mathbb{Z})$  and  $H$  a Hermitian form on  $\mathbb{C}^n$  such that  $\text{Im } H \mid \Gamma \times \Gamma = E$ .*

*Then there exists a map  $\alpha : \Gamma \rightarrow \mathbb{C}_1^* = \{z \in \mathbb{C}^* \mid |z| = 1\}$  such that*

$$\begin{aligned} \alpha(\lambda_1, \lambda_2) &= \alpha(\lambda_1)\alpha(\lambda_2) \exp(\pi\sqrt{-1}E(\lambda_1, \lambda_2)) \text{ for all } \lambda_1, \lambda_2 \in \Gamma \text{ and} \\ e_\lambda(z) &:= \alpha(\lambda) \exp(\pi H(z, \lambda) + \frac{\pi}{2}H(\lambda, \lambda)) \end{aligned}$$

*are multipliers which define a complex line bundle  $L^0 \rightarrow X$  satisfying  $c_1(L^0) = E$ .*

**Proof**

Let us set  $g_\lambda(z) = \frac{1}{2\sqrt{-1}}H(z, \lambda) + \beta_\lambda$  for any constants  $\beta_\lambda$ . Then, we have

$$\begin{aligned} &g_{\lambda_2}(z + \lambda_1) + g_{\lambda_1}(z) - g_{\lambda_1}(z + \lambda_2) - g_{\lambda_2}(z) \\ &= \frac{1}{2\sqrt{-1}}(H(z + \lambda_1, \lambda_2) + \beta_{\lambda_2} + H(z, \lambda_1) + \beta_{\lambda_1} - H(z + \lambda_2, \lambda_1) \\ &\quad - \beta_{\lambda_1} - H(z, \lambda_2) - \beta_{\lambda_2}) \\ &= \frac{1}{2\sqrt{-1}}(H(\lambda_1, \lambda_2) - H(\lambda_2, \lambda_1)) \\ &= \text{Im } H(\lambda_1, \lambda_2) \\ &= E(\lambda_1, \lambda_2) \end{aligned}$$

for all  $\lambda_1, \lambda_2 \in \Gamma$  and  $z \in \mathbb{C}^n$ .

Suppose that  $e_\lambda^0(z) := \exp(2\pi\sqrt{-1}g_\lambda(z))$  are multipliers. Then

$$e_{\lambda_2}^0(z + \lambda_1)e_{\lambda_1}^0(z) = e_{\lambda_1 + \lambda_2}^0(z)$$

for all  $\lambda_1, \lambda_2 \in \Gamma$  and  $z \in \mathbb{C}^n$ .

Then, we see that

$$\frac{1}{2\pi\sqrt{-1}}(\log e_{\lambda_2}^0(z + \lambda_1) + \log e_{\lambda_1}^0(z) - \log e_{\lambda_1 + \lambda_2}^0(z)) \in \mathbb{Z}.$$

So, from this fact,

$$\begin{aligned}
& g_{\lambda_2}(z + \lambda_1) + g_{\lambda_1}(z) - g_{\lambda_1 + \lambda_2}(z) \\
&= \frac{1}{2\sqrt{-1}}H(z + \lambda_1, \lambda_2) + \beta_{\lambda_2} + \frac{1}{2\sqrt{-1}}H(z, \lambda_1) + \beta_{\lambda_1} \\
&\quad - \frac{1}{2\sqrt{-1}}H(z, \lambda_1 + \lambda_2) - \beta_{\lambda_1 + \lambda_2} \\
&= \frac{1}{2\sqrt{-1}}H(\lambda_1, \lambda_2) + \beta_{\lambda_1} + \beta_{\lambda_2} - \beta_{\lambda_1 + \lambda_2} \in \mathbb{Z}
\end{aligned}$$

for all  $\lambda_1, \lambda_2 \in \Gamma$ .

Thus, we get

$$\frac{1}{2}H(\lambda_1, \lambda_2) + \sqrt{-1}\beta_{\lambda_1} + \sqrt{-1}\beta_{\lambda_2} - \sqrt{-1}\beta_{\lambda_1 + \lambda_2} \in \sqrt{-1}\mathbb{Z}.$$

Next, setting  $\sqrt{-1}\beta_{\lambda} = \gamma_{\lambda} + \frac{1}{4}H(\lambda, \lambda)$  for any constants  $\gamma_{\lambda}$ , we reduce the above equation to

$$\begin{aligned}
& \frac{1}{2}H(\lambda_1, \lambda_2) + \gamma_{\lambda_1} + \frac{1}{4}H(\lambda_1, \lambda_1) + \gamma_{\lambda_2} + \frac{1}{4}H(\lambda_2, \lambda_2) \\
& \quad - \gamma_{\lambda_1 + \lambda_2} - \frac{1}{4}H(\lambda_1 + \lambda_2, \lambda_1 + \lambda_2) \\
&= \frac{1}{4}(H(\lambda_1, \lambda_2) - H(\lambda_2, \lambda_1) + \gamma_{\lambda_1} + \gamma_{\lambda_2} - \gamma_{\lambda_1 + \lambda_2}) \\
&= \gamma_{\lambda_1} + \gamma_{\lambda_2} - \gamma_{\lambda_1 + \lambda_2} + \frac{\sqrt{-1}}{2}E(\lambda_1, \lambda_2) \in \sqrt{-1}\mathbb{Z}.
\end{aligned}$$

Then, from this fact, we see that  $\text{Re } \gamma_{\lambda}$  is additive in  $\Gamma$ , that is,  $\text{Re } \gamma_{\lambda} \in \text{Hom}(\Gamma, \mathbb{R})$ .

Hence,  $\text{Re } \gamma_{\lambda}$  extends to an  $\mathbb{R}$ -linear function  $\mu : \mathbb{C}^n \rightarrow \mathbb{R}$  such that  $\mu|_{\Gamma} = \text{Re } \gamma_{\lambda}$ , and there is a  $\mathbb{C}$ -linear form  $l : \mathbb{C}^n \rightarrow \mathbb{C}$  defined by  $l(z) = \mu(z) - \sqrt{-1}\mu(\sqrt{-1}z)$  with  $\text{Re } l = \mu$ .

Now, setting  $\gamma'_{\lambda} = \gamma_{\lambda} - l(z)$ ,  $\beta'_{\lambda} = \frac{1}{\sqrt{-1}}(\gamma'_{\lambda} + \frac{1}{4}H(\lambda, \lambda))$  and  $h_{\lambda}(z) = \frac{1}{2\sqrt{-1}}H(z, \lambda) + \beta'_{\lambda}$ , we calculate

$$\begin{aligned}
& h_{\lambda_2}(z + \lambda_1) + h_{\lambda_1}(z) - h_{\lambda_1 + \lambda_2}(z) \\
&= \frac{1}{2\sqrt{-1}}(H(z + \lambda_1, \lambda_2) + H(z, \lambda_1) - H(z, \lambda_1 + \lambda_2)) + \beta'_{\lambda_1} + \beta'_{\lambda_2} - \beta'_{\lambda_1 + \lambda_2} \\
&= \frac{1}{\sqrt{-1}}\left(\frac{1}{2}H(\lambda_1, \lambda_2) - \frac{1}{4}(H(\lambda_1, \lambda_2) + H(\lambda_2, \lambda_1)) + (\gamma'_{\lambda_1} + \gamma'_{\lambda_2} - \gamma'_{\lambda_1 + \lambda_2})\right) \\
&= \frac{1}{2}\text{Im } H(\lambda_1, \lambda_2) + \frac{1}{\sqrt{-1}}(\gamma'_{\lambda_1} + \gamma'_{\lambda_2} - \gamma'_{\lambda_1 + \lambda_2}) \\
&= \frac{1}{2}E(\lambda_1, \lambda_2) + \frac{1}{\sqrt{-1}}(\gamma'_{\lambda_1} + \gamma'_{\lambda_2} - \gamma'_{\lambda_1 + \lambda_2}) \in \mathbb{Z}.
\end{aligned}$$

Thus, it follows from this result that  $\exp(2\pi\sqrt{-1}h_\lambda(z))$  are multipliers.

Next, to complete our proof of theorem, it suffices to show that  $e_\lambda^0(z)$  and  $\exp(2\pi\sqrt{-1}h_\lambda(z))$  are equivalent in  $H^1(X, \mathcal{O}^*)$ . Since

$$\begin{aligned}
\exp(2\pi\sqrt{-1}h_\lambda(z)) &= \exp(\pi H(z, \lambda)) \exp(2\pi\sqrt{-1}\beta'_\lambda) \\
&= \exp(\pi H(z, \lambda)) \exp(2\pi(\gamma'_\lambda + \frac{1}{4}H(\lambda, \lambda))) \\
&= \exp(\pi H(z, \lambda)) \exp(2\pi\sqrt{-1}\beta_\lambda) \exp(-l(\lambda)) \\
&= \exp(2\pi\sqrt{-1}(\frac{1}{2\sqrt{-1}}H(z, \lambda) + \beta_\lambda)) \exp(-l(\lambda)) \\
&= e_\lambda^0(z) \exp(-l(z + \lambda)) \exp(-l(\lambda))^{-1},
\end{aligned}$$

so we obtain that  $e_\lambda^0(z)$  is equivalent to  $\exp(2\pi\sqrt{-1}h_\lambda(z))$  in  $H^1(X, \mathcal{O}^*)$ . We may assume that  $\gamma'_\lambda$  is pure imaginary.

Then, setting  $\alpha(\lambda) = \exp(2\pi\gamma'_\lambda)$ , we see that  $|\alpha(\lambda)| = 1$ .

Then, since

$$\gamma'_{\lambda_1} + \gamma'_{\lambda_2} - \gamma'_{\lambda_1 + \lambda_2} + \frac{\sqrt{-1}}{2}E(\lambda_1, \lambda_2) \in \sqrt{-1}\mathbb{Z}$$

for all  $\lambda_1, \lambda_2 \in \Gamma$ ,

$$\begin{aligned}
\frac{\alpha(\lambda_1 + \lambda_2)}{\alpha(\lambda_1)\alpha(\lambda_2)} &= \exp(2\pi(\gamma'_{\lambda_1 + \lambda_2} - \gamma'_{\lambda_1} - \gamma'_{\lambda_2})) \\
&= \exp(2\pi(\frac{\sqrt{-1}}{2}E(\lambda_1, \lambda_2) - \sqrt{-1}n)) \\
&= \exp(\pi\sqrt{-1}E(\lambda_1, \lambda_2)), \quad n \in \mathbb{Z}.
\end{aligned}$$

Therefore  $e_\lambda^0(z)$  is equivalent to  $\alpha(\lambda) \exp(\pi H(z, \lambda) + \frac{\pi}{2} H(\lambda, \lambda))$ , and hence the proof of theorem is completed.

A theta-function for  $\Gamma$  is a holomorphic function  $\theta \in H^0(\mathbb{C}^n, \mathcal{O})$  such that there exist linear polynomials  $l_\lambda(z)$  which define multipliers  $el_\lambda(z)$  satisfying

$$\theta(z + \lambda) = \theta(z)el_\lambda(z)$$

for all  $z \in \mathbb{C}^n$ .

**Definition 2.5** Let  $X = \mathbb{C}^n/\Gamma$  be a toroidal group. A multipliers is said to be a theta factor or linearizable if it is given by exponential system of linear polynomials. A line bundle  $L$  on  $X$  is a theta bundle or linearizable, if it can be given by a theta factor.

For an additive group  $\mathcal{F}$ , we denote by  $C^p(\Gamma, \mathcal{F})$  the group of  $p$ -cochains with values in  $\mathcal{F}$ ,  $Z^p(\Gamma, \mathcal{F})$  the group of  $p$ -cocycles with values in  $\mathcal{F}$  and  $B^p(\Gamma, \mathcal{F})$  the group of  $p$ -coboundaries with values in  $\mathcal{F}$ .

The following theorem was first proved by Vogt([9]).

**Theorem 2.4** *Let  $\mathbb{C}^n/\Gamma$  be a toroidal group of a cohomologically finite type. Then every complex line bundle  $L$  on  $\mathbb{C}^n/\Gamma$  is a theta bundle.*

**Proof** By theorem 2.3, we have a theta bundle  $L_0$  on  $\mathbb{C}^n/\Gamma$  which is defined by  $\alpha'(\lambda) \exp(\pi H(z, \lambda) + H(\lambda, \lambda))$  such that  $c_1(L_0) = c_1(L) = E$ , where  $E = \text{Im } H|_{\Gamma \times \Gamma}$ . Put  $L_1 := L \otimes L_0^{-1}$ . Then  $L_1$  is topologically trivial. Let  $\exp(2\pi\sqrt{-1}g_\lambda)$  ( $g_\lambda \in H^0(\mathbb{C}^n, \mathcal{O})$ ) be multipliers for  $L_1$ .

So, we see that

$$c_1(L_1)(\lambda_1, \lambda_2) = g_{\lambda_2}(z + \lambda_1) - g_{\lambda_1 + \lambda_2}(z) + g_{\lambda_1}(z) \in B^2(\Gamma, \mathbb{Z})$$

for all  $\lambda_1, \lambda_2 \in \Gamma$

This means that there exist  $\alpha_\lambda \in C^1(\Gamma, \mathbb{Z})$  such that

$$g_{\lambda_2}(z + \lambda_1) - g_{\lambda_1 + \lambda_2}(z) + g_{\lambda_1}(z) = \alpha_{\lambda_2} - \alpha_{\lambda_1 + \lambda_2} + \alpha_{\lambda_1}.$$

Next, replacing  $g_\lambda$  by  $g_\lambda - \alpha_\lambda$ , then we get



$$g_{\lambda_2}(z + \lambda_1) - g_{\lambda_1 + \lambda_2}(z) + g_{\lambda_1}(z) = 0.$$

Thus, from the above equation, we see that  $g_\lambda \in Z^1(\Gamma, \mathcal{H})$ , where  $\mathcal{H} = H^0(\mathbb{C}^n, \mathcal{O})$ .

So, according to our assumption that  $\mathbb{C}^n/\Gamma$  is a cohomologically finite type, the map

$$H^1(\mathbb{C}^n/\Gamma, \mathbb{C}) \longrightarrow H^1(\mathbb{C}^n/\Gamma, \mathcal{O}) \quad \text{is surjective}$$

and also the map

$$H^1(\Gamma, \mathbb{C}) \longrightarrow H^1(\Gamma, \mathcal{H}) \quad \text{is surjective.}$$

Then, there exist  $c_\lambda \in Z^1(\Gamma, \mathbb{C})$  such that

$$g_\lambda(z) - c_\lambda(z) = h(z + \lambda) - h(z), \quad \text{for some } h \in C^0(\Gamma, \mathcal{H}).$$

From the above equation, we get

$$\exp(2\pi\sqrt{-1}g_\lambda(z)) = \exp(2\pi\sqrt{-1}c_\lambda(z)) \exp(h(z + \lambda))\exp(h(z))^{-1}.$$

This implies that  $\exp(2\pi\sqrt{-1}c_\lambda(z))$  are the multipliers for  $L_1$ . Since the line bundle  $L_1$  is topologically trivial, so  $c_\lambda \in \text{Hom}(\Gamma, \mathbb{C})$ . Therefore, there exists a  $\mathbb{C}$ -linear form  $\varphi : \mathbb{C}^n \longrightarrow \mathbb{C}$  satisfying

$$\text{Im } \varphi \mid \Gamma = \text{Im } c_\lambda.$$

So we get

$$\exp 2\pi\sqrt{-1}(c_\lambda - \varphi(\lambda)) = \exp 2\pi\sqrt{-1}(c_\lambda(z)) \exp(2\pi\sqrt{-1}(-\varphi(z + \lambda) + \varphi(z))).$$

This then means that  $\exp 2\pi\sqrt{-1}(c_\lambda - \varphi(\lambda))$  are also the multipliers for  $L_1$ .

On the other hand, we see  $c_\lambda - \varphi(\lambda) \in \mathbb{R}$  since  $\text{Im}(c_\lambda - \varphi(\lambda)) = 0$  on  $\Gamma$ .

Setting  $\exp 2\pi\sqrt{-1}(c_\lambda - \varphi(\lambda)) = \psi(\lambda)$  and  $\alpha(\lambda) = \psi(\lambda)\alpha'(\lambda)$ , since  $L_1 := L \otimes L_0^{-1}$ , then  $\alpha(\lambda) \exp(\pi H(z, \lambda) + \frac{\pi}{2} H(\lambda, \lambda))$  are the multipliers for

$L$ . Therefore, it follows from this result that  $L$  is represented by linear polynomial, and hence we complete the proof of theorem.

For a proof of main theorem, we need the following notations.

Let  $\mathbb{C}^n/\Gamma$  be a toroidal group of type  $q$ . After a linear change of coordinates of  $\mathbb{C}^n$ , we see  $\mathbb{C}^n/\Gamma$  has a period matrix of the form  $P = [I_n, V]$ , where  $I_n = [e_1, \dots, e_n]$  is the  $n \times n$  unit matrix and  $V = [v_{ij}; 1 \leq i \leq n, 1 \leq j \leq q] = [v_1, \dots, v_q]$  is a  $n \times q$  matrix. Put  $V_1 = [v_{ij}; 1 \leq i, j \leq q]$ , and  $V_2 = [v_{ij}; q+1 \leq i \leq n, 1 \leq j \leq q]$ . We may assume  $\det(\operatorname{Im} V_1) \neq 0$ . We put  $v_i = \sqrt{-1}e_i$  for  $q+1 \leq i \leq n$ , and  $\beta_i = \operatorname{Im} v_i$  for  $1 \leq i \leq n$ . Then  $\beta_1, \dots, \beta_n$  are linearly independent over  $\mathbb{C}$ . Put

$$z = z_1\beta_1 + \dots + z_n\beta_n.$$

Then we have  $\mathbb{C}_\Gamma = \mathbb{C}\{\beta_1, \dots, \beta_n\}$ .

We have the following(cf. [10])

**Lemma 2.1** *Let  $L$  be a topologically trivial line bundle on a toroidal group  $X = \mathbb{C}^n/\Gamma$  of cohomologically finite type. If there exists  $s \in H^0(X, \mathcal{O}(L))$  which is not identically zero, then  $L$  is analytically trivial.*

**Proof**

By Theorem 2.4,  $L$  is defined by multipliers

$$\alpha(\lambda) \exp(\pi H(z, \lambda) + \frac{\pi}{2} H(\lambda, \lambda)),$$

where  $\operatorname{Im} H|_{\Gamma \times \Gamma} = c_1(L)$ .

Since  $c_1(L) = 0$ , we may assume  $H = 0$ . Then the holomorphic section  $s(z)$  is a holomorphic function on  $\mathbb{C}^n$  satisfying

$$s(z + \lambda) = \alpha(\lambda)s(z), \text{ for } z \in \mathbb{C}^n \text{ and } \lambda \in \Gamma.$$

Hence

$$|s(z + \lambda)| = |s(z)|, \text{ for } z \in \mathbb{C}^n \text{ and } \lambda \in \Gamma.$$

Then  $|s(z)|$  is bounded on the maximal compact subgroup  $\mathbb{R}_\Gamma/\Gamma$  of  $\mathbb{C}^n/\Gamma$ . Hence  $s(z)$  is a bounded holomorphic function on  $\mathbb{C}_\Gamma$ . Then  $s(z) = \text{constant}$  on  $\mathbb{C}_\Gamma$ .

Let  $\mathbb{C}^n/\Gamma$  has a period matrix of the form  $P = [I_n, V]$ . Then  $s(z)$  is holomorphic function of  $z_{q+1}, \dots, z_n$ . Put

$z' = {}^t(z_1, \dots, z_q) \in \mathbb{C}^q$ ,  $z'' = {}^t(z_{q+1}, \dots, z_n) \in \mathbb{C}^{n-q}$ ,  $\pi'(z) = z'$  and  $\pi''(z) = z''$ , for  $z \in \mathbb{C}^n$ .

For any vectors  $\mu_1, \dots, \mu_r$  in  $\mathbb{C}^n$  and matrix  $M = [\mu_1, \dots, \mu_r]$ , we write  $M' = \pi' M = [\mu'_1, \dots, \mu'_r]$ . Similarly we write  $M''$ .

We have a holomorphic function  $\hat{s}(z'')$  on  $\mathbb{C}^{n-q}$  such that  $s(z) = \hat{s}(\pi''(z))$ .

Suppose there exists  $z^0 \in \mathbb{C}^n$  such that  $s(z^0) = 0$ . We may assume  $z^0 = 0$ . Then  $s(\lambda) = \hat{s}(\pi''(\lambda)) = 0$ , for all  $\lambda \in \Gamma$ . Put

$$V = \alpha + \sqrt{-1}\beta,$$

Then

$$P'' = [-\beta''\beta'^{-1}, I_{n-q}, \alpha'' - \beta''\beta'^{-1}\alpha'],$$

where  $I_{n-q}$  is the identity matrix of degree  $n - q$ . Put

$$\hat{P}'' = [I_{n-q}, R],$$

where  $R = [-\beta''\beta'^{-1}, \alpha'' - \beta''\beta'^{-1}\alpha']$ .

Since  $\mathbb{C}^n/\Gamma$  is toroidal,  ${}^t\sigma R \notin {}^t\mathbb{Z}^{2q}$  for any  $\sigma \neq 0 \in \mathbb{Z}^{n-q}$ .

Hence  $\mathbb{Z}\{P''\}$  is dense in  $\mathbb{R}^{n-q}$ .

Since

$$s(\lambda) = \hat{s}(\lambda'') = 0 \text{ for all } \lambda \in \Gamma, \text{ and } \lambda'' \in \mathbb{Z}\{P''\},$$

$$\hat{s}(x) = 0 \text{ for all } x \in \mathbb{R}^{n-q},$$

then

$$\hat{s}(z'') = 0 \text{ for all } z'' \in \mathbb{C}^{n-q}. \text{ Hence } s(z) = 0 \text{ for all } z \in \mathbb{C}^n.$$

But this is a contradiction. Hence the lemma is proved.

Now we return to prove theorem 2.2.

**Proof** Let  $f$  be a meromorphic function on  $\mathbb{C}^n$  with  $\Gamma \subset G(f)$ .

Then, there exist  $p, q \in H^0(\mathbb{C}^n, \mathcal{O})$  with  $f = p/q$  and  $(p, q) = 1$ . Moreover there exist linear polynomials  $l_\lambda(z)$  such that

$$p(z + \lambda) = el_\lambda(z)p(z) \text{ and } q(z + \lambda) = el_\lambda(z)q(z),$$

for all  $z \in \mathbb{C}^n$ . By the assumption  $NS(X) = 0$ . Hence  $p(z)$  and  $q(z)$  are the holomorphic sections of topologically trivial line bundle on  $X$ . Since  $p(z)$  and  $q(z)$  are not identically zero, these are the sections of analytically trivial line bundle. Since  $X$  is toroidal  $p(z)$  and  $q(z)$  are constant. Hence there are no non-constant meromorphic functions on  $X$  and theorem is proved.

### 3 Existence of non-constant meromorphic functions on $X$

In this section we shall discuss the example given by Abe and Kopfermann. They gave an example [2] of a non-compact toroidal group which has only constants as meromorphic functions. It is a toroidal group  $X = \mathbb{C}^n/\Gamma$ , where  $\Gamma = \mathbb{Z}\{P\}$  and

$$P = \begin{bmatrix} 1 & 0 & 0 & i & \sqrt{2}i \\ 0 & 1 & 0 & \sqrt{3}i & \sqrt{5}i \\ 0 & 0 & 1 & \sqrt{7}i & i \end{bmatrix}$$

They asserted that all meromorphic functions on  $X$  are constant. However, by using the recent results of Umeno [8], we can see that there exist non-constant meromorphic functions on  $X$ .

Next, for later use, we shall state the following results proved in [8].

**Theorem 3.1** ([8], Theorem 3.1) *Let  $X = \mathbb{C}^n/\Gamma$  be a toroidal group of type  $q$ , with a period matrix of the form  $P = [\lambda_1, \dots, \lambda_{n+q}] = [I_n, V]$ .*

(1) *If  $\mathbb{C}^n/\Gamma$  is a quasi-Abelian variety with an ample Riemann form  $H$ , then  $E := \text{Im}H \mid \Gamma \times \Gamma$  satisfies the following conditions:*

$$R1 = {}^tV E_1 V + {}^tE_2 V - {}^tV E_2 + E_3 = 0$$

$$R2 = \frac{\sqrt{-1}}{2}({}^t\bar{V} E_1 V + {}^tE_2 V - {}^t\bar{V} E_2 + E_3) > 0,$$

where  $E = \begin{bmatrix} E_1 & E_2 \\ -{}^tE_2 & E_3 \end{bmatrix}$ ,  $E_1 \in \mathbb{Z}^{n+n}$ , and  $E_3 \in \mathbb{Z}^{q \times q}$ .

(2) *Conversely, if we have a  $\mathbb{Z}$ -valued skew-symmetric matrix  $E = [E(\lambda_i, \lambda_j); 1 \leq i, j \leq n+q] \in \mathbb{Z}^{(n+q) \times (n+q)}$ , which satisfies  $R1$  and  $R2$ , then  $X = \mathbb{C}^n/\Gamma$  is a quasi-Abelian variety with an ample Riemann form  $H$  satisfying  $\text{Im}H \mid \Gamma \times \Gamma = E$ .*

The following result is about a period matrix which characterize a quasi-Abelian variety.

**Theorem 3.2** ([8], Theorem 3.4) *Let  $X = \mathbb{C}^n/\Gamma$  be a toroidal group. Then  $X = \mathbb{C}^n/\Gamma$  is a quasi-Abelian variety of type  $q$  if and only if there exist a*

basis  $\lambda_1, \dots, \lambda_{n+q}$  for  $\Gamma$  and a complex basis  $e_1, \dots, e_n$  for  $\mathbb{C}^n$  such that the period matrix

$$P = [\lambda_1, \dots, \lambda_{n+q}] = [\Delta(q, n), W],$$

where  $\Delta(q, n) := [\delta_1 e_1, \dots, \delta_q e_q, e_{q+1}, \dots, e_n] \in \mathbb{Z}^{n+n}$ , with positive integers

$\delta_1 | \delta_2 | \dots | \delta_q$  and  $W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \in \mathbb{C}^{n \times q}$  satisfying  $W_1 \in \mathbb{C}^{q \times q}$  is symmetric and  $\text{Im} W_1 > 0$ .

To do our goal, we have only to find  $E$  satisfying the conditions R1 and R2 of theorem 3.1. Then, we have the following:

**Proposition 3.1** *Let  $\mathbb{C}^3/\Gamma$ , where  $\Gamma = \mathbb{Z}\{P\}$  be a toroidal group of type 2*

*with a period matrix of the form  $P = [I_3, V] = \begin{bmatrix} 1 & 0 & 0 & i & \sqrt{2}i \\ 0 & 1 & 0 & \sqrt{3}i & \sqrt{5}i \\ 0 & 0 & 1 & \sqrt{7}i & i \end{bmatrix}$ .*

*Then we get a  $\mathbb{Z}$ -valued skew-symmetric form  $E$  such that satisfies*

$${}^t V E_1 V + {}^t E_2 V - {}^t V E_2 + E_3 = 0 \quad (1)$$

$$\frac{\sqrt{-1}}{2} ({}^t \bar{V} E_1 V + {}^t E_2 V - {}^t \bar{V} E_2 + E_3) > 0, \quad (2)$$

**Proof** We first recall the peroid matrix of the form  $P = [I_3, V] = \begin{bmatrix} 1 & 0 & 0 & i & \sqrt{2}i \\ 0 & 1 & 0 & \sqrt{3}i & \sqrt{5}i \\ 0 & 0 & 1 & \sqrt{7}i & i \end{bmatrix}$ .

Then, we set  $E$  as the following form:  $E = \begin{bmatrix} E_1 & E_2 \\ -{}^t E_2 & E_3 \end{bmatrix}$ , where

$$E_1 = \begin{bmatrix} 0 & -p & -a \\ p & 0 & -b \\ a & b & 0 \end{bmatrix}, E_2 = \begin{bmatrix} -e & -h \\ -f & -i \\ -g & -j \end{bmatrix}, \text{ and } E_3 = \begin{bmatrix} 0 & -c \\ c & 0 \end{bmatrix}.$$

We note  $E$  is a  $\mathbb{Z}$ -valued skew-symmetric form.

Substituting  $E$  into R1 and R2, then we have  $R1 = \begin{bmatrix} 0 & r \\ -r & 0 \end{bmatrix}$ , where

$$r = (a - \sqrt{14}a + \sqrt{3}b - \sqrt{35}b - c + \sqrt{5}p - \sqrt{6}p) + i(-\sqrt{2}e - \sqrt{5}f - g + h + \sqrt{3}i + \sqrt{7}j), \text{ where } a, b, \dots, p \in \mathbb{Z}$$

$$\text{and } R2 = \begin{bmatrix} \sqrt{7}g & g \\ g & \sqrt{2}g \end{bmatrix}.$$

Then, for satisfying the conditions (1) and (2),  $a = b = c = p = 0, e = f = i = j = 0$  and  $g = h$ , where  $g > 0$ .

$$\text{Therefore, we get } E = \begin{bmatrix} 0 & 0 & 0 & 0 & -g \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -g & 0 \\ 0 & 0 & g & 0 & 0 \\ g & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ where } g(> 0) \in \mathbb{Z} \text{ which}$$

satisfies the conditions (1) and (2). The proof is completed.

Then, the above proposition implies that  $\mathbb{C}^3/\Gamma$  is a quasi-Abelian variety from Theorem 3.1.

After a linear change of coordinates, by setting

$$\lambda_1 = ge'_1 = e_3, \lambda_2 = ge'_2 = e_1, \lambda_3 = e'_3 = e_2, \lambda_4 = v_1, \lambda_5 = v_2,$$

$$\text{we get an alternating form } [E(\lambda_i, \lambda_j); 1 \leq i, j \leq 5] = \begin{bmatrix} 0 & 0 & -\Delta(g) \\ 0 & 0 & 0 \\ \Delta(g) & 0 & 0 \end{bmatrix},$$

where  $\Delta(g) = \text{diag}(g, g)$ .

Thus, it follows from the same way that we get the period matrix

$$P' = [\Delta(g), V'] = \begin{bmatrix} g & 0 & 0 & \sqrt{7}gi & gi \\ 0 & g & 0 & gi & \sqrt{2}gi \\ 0 & 0 & 1 & \sqrt{3}i & \sqrt{5}i \end{bmatrix}$$

from the period matrix  $P$ , where  $V'$  is a representation of  $V$  with respect to a new basis  $e'_1, e'_2, e'_3$  for  $\mathbb{C}^3$ . Then

$$V' = \begin{bmatrix} V'_1 \\ V'_2 \end{bmatrix} = \begin{bmatrix} \sqrt{7}gi & gi \\ gi & \sqrt{2}gi \\ \sqrt{3}i & \sqrt{5}i \end{bmatrix}, \text{ where } V'_1 \in \mathbb{C}^{2 \times 2} \text{ and } g(> 0) \in \mathbb{Z}$$

satisfies that  $V'_1$  is symmetric and  $\text{Im}V'_1$  is positive definite.

Hence  $\mathbb{C}^3/\Gamma'$ , where  $\Gamma' = \mathbb{Z}\{P'\}$  is a quasi-Abelian variety of type 2 from the Theorem 3.2.

Then, to make sure the result, we project the period matrix  $P'$  to  $\mathbb{C}^2$ . It suffices to show that the 2-dimensional torus group generated by  $P'^*$  is an abelian variety. Here, the period matrix  $P'^*$  is of the form

$$\begin{bmatrix} g & 0 & \sqrt{7}g_i & g_i \\ 0 & g & g_i & \sqrt{2}g_i \end{bmatrix} = [\Delta(g), Z].$$

Then  $Z$  is symmetric and  $\operatorname{Im} Z$  is positive definite.

Therefore, from the Riemann conditions III [5],  $\mathbb{C}^2/\Gamma'^*$ , where  $\Gamma'^* = \mathbb{Z}\{P'^*\}$  is an abelian variety.

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